

## Mathematical Methods in Physics HW4

1. Consider anti-self-adjoint matrices. Prove that their eigenvalues are purely imaginary. Find a couple of 2x2 examples of a matrices which are both anti-self-adjoint as well as unitary, and verify that their eigenvalues are imaginary (no real part).

First of all let's prove that for anti-self-adjoint matrices,  $(x, Mx)$  is purely imaginary.

$$(x, Mx) = (M^\dagger x, x) = (-Mx, x) = -(Mx, x) = -(x, Mx)^* \text{ but if } a = -a^* \text{ then } a \text{ is purely imaginary.}$$

Now let's prove that the eigenvalues of an anti-self-adjoint matrix are purely imaginary.

$$Mx = \lambda x \Rightarrow (x, Mx) = (x, \lambda x) = \lambda(x, x) \Rightarrow \lambda = \frac{(x, Mx)}{\|x\|^2} \text{ which is purely imaginary since } (x, Mx) \text{ is purely imaginary and } \|x\|^2 \text{ is real.}$$

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow M = -M^\dagger = -\tilde{M} \text{ and } MM^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det[M - \lambda I] = (-\lambda)^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$$

$$M' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \Rightarrow M' = -M'^\dagger \text{ and } M'M'^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det[M' - \lambda I] = (-\lambda)^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$$

2. Recall the three results for isometric matrices:

- a)  $UU^\dagger = I$
- b)  $(Ux, Uy) = (x, y)$  for all  $x$  and  $y$ .
- c)  $\|Ux\| = \|x\|$  for all  $x$ .

For which we showed that (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c)  $\Rightarrow$  (a) in class. Show that (c)  $\Rightarrow$  (b)  $\Rightarrow$  (a)  $\Rightarrow$  (c) as well.

$$\text{For (c) } \Rightarrow \text{ (b): If } \|Ux\| = \|x\| \Rightarrow \sqrt{(Ux, Ux)} = \sqrt{(x, x)} \Rightarrow (Ux, Ux) = (x, x) \text{ for all } x.$$

First take  $x = y + z$  in which case  $(U[y + z], U[y + z]) = ([y + z], [y + z])$  gives:

$$(Uy, Uy) + (Uy, Uz) + (Uz, Uy) + (Uz, Uz) = (y, y) + (y, z) + (z, y) + (z, z)$$

Noting that  $(Ux, Ux) = (x, x)$  for all  $x$  this leaves:

$$(Uy, Uz) + (Uz, Uy) = (y, z) + (z, y) \Rightarrow (Uy, Uz) + (Uy, Uz)^* = (y, z) + (y, z)^*$$

But this implies that  $2\text{Re}[(Uy, Uz)] = 2\text{Re}[(y, z)]$  or  $\text{Re}[(Uy, Uz)] = \text{Re}[(y, z)]$  for all  $y, z$ .

Now take  $x = y + iz$  in which case  $(U[y + iz], U[y + iz]) = ([y + iz], [y + iz])$  gives:

$$(Uy, Uy) + (Uy, iUz) + (iUz, Uy) + (iUz, iUz) = (y, y) + (y, iz) + (iz, y) + (iz, iz)$$

$$(Uy, Uy) + i(Uy, Uz) - i(Uz, Uy) + (Uz, Uz) = (y, y) + i(y, z) - i(z, y) + (z, z)$$

Noting that  $(Ux, Ux) = (x, x)$  for all  $x$  this leaves:

$$i(Uy, Uz) - i(Uz, Uy) = i(y, z) - i(z, y) \Rightarrow i(Uy, Uz) - i(Uy, Uz)^* = i(y, z) - i(y, z)^*$$

But this implies that  $2\text{Im}[(Uy, Uz)] = 2\text{Im}[(y, z)]$  or  $\text{Im}[(Uy, Uz)] = \text{Im}[(y, z)]$  for all  $y, z$ .

Then  $(Uy, Uz) = (y, z)$  for all  $y, z$ .

For (b)  $\Rightarrow$  (a): If  $(Ux, Uy) = (x, y)$  for all  $x, y \Rightarrow (U^+Ux, y) = (x, y) \Rightarrow ([U^+U - I]x, y) = 0$  for all  $x, y$ . But this implies that  $U^+U - I = 0$  then  $U^+U = I$ .

For (a)  $\Rightarrow$  (c): If  $U^+U = I$  then  $\|Ux\| = \sqrt{(Ux, Ux)} = \sqrt{(U^+Ux, x)} = \sqrt{(x, x)} = \|x\|$ .

3. Consider the following matrices. Prove that each is normal. Then identify further properties of each, i.e. Hermitian, unitary, symmetric, orthogonal. Find the eigenvalues of each matrix, then find the corresponding eigenvectors. Then find all of the matrices which diagonalize these, and identify the type of matrices that they are, i.e. Hermitian, unitary, symmetric, orthogonal. And finally, using these matrices show the diagonal form of the original matrices.

a)  $M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$M$  is a symmetric matrix, i.e.  $\tilde{M} = M$ , hence normal. The eigenvalues are given by

$$\det[M - \lambda I] = 0 = (1 - \lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4 \\ = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \Rightarrow \lambda_1 = 3, \lambda_2 = -1$$

The eigenvectors associated with each are:

$$\lambda_1 = 3 \Rightarrow Mx_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \end{pmatrix} \Rightarrow \begin{matrix} a + 2b = 3a \\ 2a + b = 3b \end{matrix} \Rightarrow \begin{matrix} 2b = 2a \\ 2a = 2b \end{matrix} \Rightarrow a = b \Rightarrow \hat{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\lambda_2 = -1 \Rightarrow Mx_2 = M \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix} \Rightarrow \begin{matrix} a + 2b = -a \\ 2a + b = -b \end{matrix} \Rightarrow \begin{matrix} 2b = -2a \\ 2a = -2b \end{matrix} \Rightarrow a = -b \Rightarrow \hat{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \text{ } P \text{ is symmetric and orthogonal } \Rightarrow M_{diag} = P^{-1}MP = \tilde{P}MP = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

$$P' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \text{ } P' \text{ is antisymmetric and orthogonal } \Rightarrow M'_{diag} = P'^{-1}MP' = \tilde{P}'M = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$b) N = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$N$  is an antisymmetric matrix and is orthogonal, i.e.  $\tilde{N} = -N$  and  $\tilde{N}N = I$ , hence normal. The eigenvalues are given by

$$\det[N - \lambda I] = 0 = (-\lambda)^2 + 1 = \lambda^2 + 1$$

$$\lambda^2 = -1 \Rightarrow \lambda_1 = i, \lambda_2 = -i$$

The eigenvectors associated with each are:

$$\lambda_1 = i \Rightarrow Nx_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ia \\ -ib \end{pmatrix} \Rightarrow \begin{matrix} b = ia \\ -a = ib \end{matrix} \Rightarrow b = ia \Rightarrow \hat{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\lambda_2 = -i \Rightarrow Nx_2 = N \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ia \\ -ib \end{pmatrix} \Rightarrow \begin{matrix} b = -ia \\ -a = -ib \end{matrix} \Rightarrow b = -ia \Rightarrow \hat{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \end{pmatrix} P \text{ is unitary} \Rightarrow N_{diag} = P^{-1}NP = \tilde{P}NP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$P' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \end{pmatrix} P' \text{ is unitary} \Rightarrow N'_{diag} = P'^{-1}NP' = \tilde{P}'NP' = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

4. Consider the following statement: For a Hermitian matrix that is fully degenerate, i.e. all of its eigenvalues are the same, then the matrix is necessarily diagonal. If this is true, prove it. If not, find a counter example.

First of all, recall from class that any Hermitian matrix is diagonalizable by a unitary similarity transformation. So we know that in at least one basis, the matrix is diagonalizable. Now since it has the same eigenvalues, it takes the diagonal form  $M = \begin{pmatrix} a & 0 & \dots \\ 0 & a & \dots \\ \vdots & & \ddots \end{pmatrix} = a \begin{pmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & & \ddots \end{pmatrix} = aI$ . But now let's see if it is possible to transform this via a similarity transformation:

$$M \rightarrow M' = U^{-1}MU = U^{-1}kIU = kU^{-1}U = kI = M$$

so it cannot be transformed away from this diagonal form. Since we can always get it diagonal, it must always be diagonal.